A Fluctuation Limit Theorem of Branching Processes with Immigration and Statistical Applications

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Abstract. We prove a general fluctuation limit theorem for Galton-Watson branching processes with immigration. The limit is a time-inhomogeneous OU type process driven by a spectrally positive Lévy process. As applications of this result, we obtain some asymptotic estimates for the conditional least squares estimators of the means and variances of the offspring and immigration distributions.

Key words. Branching process with immigration; Ornstein-Uhlenbeck type process; one-sided stable process; fluctuation limit; Poisson random measure; conditional least squares estimator

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1 Introduction

Consider the *Galton-Watson branching process with immigration*, $\{y(k) : k = 1, 2, \cdots\}$, defined by

$$y(k) = \sum_{j=1}^{y(k-1)} \xi(k, j) + \eta(k), \quad k \geq 1, \qquad y(0) = 0, \tag{1.1}$$

where $\{\xi(k, j) : k, j = 1, 2, \cdots\}$ and $\{\eta(k) : k = 1, 2, \cdots\}$ are two independent families of i.i.d. random variables taking values in $\mathbb{N} := \{0, 1, 2, \cdots\}$. The distribution of $\xi(k, j)$ is called the offspring distribution and the distribution of $\eta(k)$ is called the immigration distribution. Let $g(\cdot)$ and $h(\cdot)$ be the generating functions of $\xi(k, j)$ and $\eta(k)$, respectively. It is easy to see that $\{y(k)\}$ is a discrete-time Markov chain with values in $\mathbb{N}$ and one-step transition matrix $P(i, j)$ given by

$$\sum_{j=0}^{\infty} P(i, j)s^j = g(s)^i h(s), \qquad i \in \mathbb{N}, \ 0 \leq s \leq 1. \tag{1.2}$$

For simplicity, we also call $\{y(k)\}$ a *GWI-process with parameters* (g, h). Assume that the offspring mean $m := g'(1)$ is finite. The cases $m > 1$, $m = 1$ and $m < 1$ are referred to respectively as *supercritical, critical, and subcritical*. A sequence of GWI-processes $\{y_n(\cdot)\}$ with (g_n, h_n) is said to be *nearly critical* if $m_n := g_n'(1)$ converges to 1 as n tends to ∞.

The estimation problem for the offspring and immigration parameters in the GWI-process has been extensively studied; see Heyde and Seneta [8, 9], Wei and Winnicki [25] and the references therein. It is well known that the conditional least squares estimators (CLSE), first obtained by

Klimko and Nelson [14], can be used to estimate the offspring mean on the basis of the observing information on $\{y(k)\}$; see also [8] and [25] for other closely related estimators. In the non-critical case, the CLSE of the offspring mean is consistent and asymptotically normal (see [14], [24, 25]). However, it was shown by [22] and [24] that in the critical or nearly critical case the CLSE is not asymptotically normal. In fact, when the process is nearly critical and the offspring variance tends to a positive real number, Sriram [22] gave the weak convergence of GWI-processes to the branching diffusion with immigration. As a result, the above CLSE of the offspring mean has the asymptotic distribution which is expressed in terms of the limit process and the normalizing factor is n. Motivated by the similar statistical application, Ispány *et al.* [11] have recently obtained a fluctuation limit theorem for the nearly critical processes where the offspring variances tend to 0. Such limit is a time-inhomogeneous Ornstein-Uhlenbeck (OU) processes driven by a Wiener process. As a consequence, they proved the asymptotic normality of CLSE of the offspring mean with normalizing factor $n^{3/2}$. Obviously, the asymptotic behavior of the CLSE in the critical or nearly critical case is closely related to the limit theorems of the GWI-processes.

The main objective of this paper is to give a general fluctuation limit theorem and its applications for processes that allow the offspring and immigration distributions to have infinite variances. Fluctuation limits for branching models with immigration have been investigated by Dawson and Li [4], Ispány *et al.* [11], Li [18] and Li and Ma [20]; see also Dawson *et al.* [3] for the type of limits in the measure-valued setting. In the present paper, we shall consider a sequence of nearly critical GWI-processes $\{y_n(\cdot)\}$ with (g_n, h_n) satisfying a set of conditions similar to that of [18]. Let us define the sequence $Y_n(t) = y_n([nt])$ and consider the rescaled centralized process $Z_n(\cdot) = c_n^{-1}(Y_n(\cdot) - E[Y_n(\cdot)])$ with certain sequence of positive constants c_n. It turns out that $Z_n(\cdot)$ converges to a time-inhomogeneous OU type process driven by a spectrally positive Lévy process (Theorems 2.2). Based on this fluctuation limit, we show that non-degenerate limit laws still exist for the above CLSE estimates of means (Theorem 3.2). Of special interest is the case when the offspring and immigration distributions belong to the domain of attraction of a stable law with exponent α $(1 < \alpha \leq 2)$. For simplicity, suppose that

$$g_n(s) = s + \frac{\gamma}{n}(1-s)^\alpha \quad \text{and} \quad h_n(s) = s + \varpi(1-s)^\alpha,$$

where $0 < \gamma, \varpi \leq 1/\alpha$. Note that for $1 < \alpha < 2$, g_n has infinite variance but its heavy-tailed effect weakens as $n \to \infty$; for $\alpha = 2$, the offspring variance is $2\gamma/n$ and tends to 0. Then $Z_n(\cdot)$ with $c_n = n^{1/\alpha}$ converges to a OU type process driven by a α-stable process (Corollary 2.3). As a consequence, the CLSE of the offspring mean is asymptotic to a α-stable distribution and the normalizing factor is $n^{\frac{2\alpha-1}{\alpha}}$ (Corollary 3.1). As mentioned above, the estimation for the offspring mean in GWI-process have been systematically studied by [24, 25], [22] and [11], provided that the offspring variances are finite. Our results can be regarded as an attempt in the case when the above assumption fail to hold.

Another interesting case, related to our limit theorem, is that the offspring variances are finite and tend to 0, but the offspring distributions do not satisfy the Lindeberg conditions required in [11]. Then the resulting fluctuation limit $Z(\cdot)$ is a OU type process with positive jumps instead of OU diffusion (Corollary 2.2). In this case, it is also possible to consider the CLSE estimates for the offspring and immigration variances. We show that the CLSE of the offspring variance is consistent and its asymptotic distribution (with normalizing factor n) is expressed in terms of the jumps of $Z(\cdot)$, while the CLSE of the immigration variance is not consistent (Theorme 3.3). However, if we return to the case of [11] (see also Example 2.1), the above asymptotic distribution is degenerate to 0 and the above immigration variance estimator becomes consistent (Remark 3.1). Hence, in this case, by adding certain conditions on fourth moments we further prove that these estimators of the offspring and immigration variances are asymptotically normal with the normalizing factors $n^{3/2}$ and $n^{1/2}$, respectively (Theorem 3.4). This result also contrasts with

the critical-mean and positive-variance case of Winnicki [26], in which the CLSE of the offspring variance is not asymptotically normal, although it has another limit law with the normalizing factor $n^{1/2}$.

The remainder of this paper is organized as follows. The main limit theorems and some examples will be given in section 2. In Section 3 we obtain some asymptotic estimates for the statistics of the GWI-process, as applications of our limit theorems. Section 4 is devoted to the proofs of Theorem 2.1-2.2 and Theorem 3.1-3.4.

Notation. Let $\mathbb{R}_+ = [0,\infty)$. For $x \in \mathbb{R}$, set $\chi(x) = (1 \wedge x) \vee (-1)$. " $\xrightarrow{p}$ " and " $\xrightarrow{d}$ " denote the convergence of random variables in probability and convergence in distribution, respectively. We also make the convention that $\int_r^t = -\int_t^r = \int_{(r,t]}$ and $\int_r^\infty = \int_{(r,\infty)}$ for $r \le t \in \mathbb{R}$.

2 Limit theorems and examples

Let us consider a sequence of GWI-processes $y_n(\cdot)$ with parameters (g_n, h_n). A realization of $y_n(\cdot)$ is defined by

$$y_n(k) = \sum_{j=1}^{y_n(k-1)} \xi_n(k,j) + \eta_n(k), \quad k \ge 1, \qquad y_n(0) = 0, \tag{2.1}$$

where $\{\xi_n(k,j)\}$ and $\{\eta_n(k)\}$ are given as in (1.1), but depend on the index n. Also, g_n and h_n are the generating functions of $\xi_n(k,j)$ and $\eta_n(k)$. Now introduce the sequence

$$Y_n(t) := y_n([nt]), \qquad t \ge 0,$$

where $[nt]$ denotes the integer-part of nt, and $Y_n'(t) := \sum_{k=1}^{[nt]} \eta_n(k)$. We first prove a limit theorem for the sequence $(Y_n(\cdot), Y_n'(\cdot))$. Such theorem is the modification of Theorem 2.1 in [19]. Let $\{b_n\}$ be a sequence of positive numbers such that $b_n \to \infty$ as $n \to \infty$. For $0 \le \lambda \le b_n$, set

$$R_n(\lambda) = nb_n[(1-\lambda/b_n) - g_n(1-\lambda/b_n)] \tag{2.2}$$

and

$$F_n(\lambda) = n[1 - h_n(1-\lambda/b_n)]. \tag{2.3}$$

Consider the following set of conditions:

(A) The sequence $\{R_n\}$ is uniformly Lipschitz on each bounded interval and converges to a continuous function as $n \to \infty$;

(B) The sequence $\{F_n\}$ converges to a continuous function as $n \to \infty$.

Lemma 2.1 *Under condition (A), the limit function R of $\{R_n\}$ has representation*

$$R(\lambda) = c\lambda - \theta\lambda^2 - \int_0^\infty (e^{-\lambda u} - 1 + \lambda u)\,\Lambda_1(du), \tag{2.4}$$

where $c \in \mathbb{R}$, $\theta \ge 0$, and $\Lambda_1(du)$ is a σ-finite measure on $(0,\infty)$ with $\int_0^\infty (u \wedge u^2)\Lambda_1(du) < \infty$.

Lemma 2.2 *Under condition (B), the limit function F of $\{F_n\}$ has representation*

$$F(\lambda) = d\lambda + \int_0^\infty (1 - e^{-\lambda u})\, \Lambda_2(du), \tag{2.5}$$

where $d \geq 0$, and $\Lambda_2(dz)$ is a σ-finite measure on $(0, \infty)$ with $\int_0^\infty (1 \wedge u)\, \Lambda_2(du) < \infty$.

Theorem 2.1 *Suppose that (A) and (B) are satisfied. Then $\big(Y_n(\cdot)/b_n, Y_n'(\cdot)/b_n\big)$ converges in distribution on $D([0,\infty), \mathbb{R}_+^2)$ to a two-dimensional non-negative Markov process $\big(Y(\cdot), Y'(\cdot)\big)$ with initial value $(0,0)$ and transition semigroup $(P_t)_{t\geq 0}$ given by*

$$\int_{\mathbb{R}_+^2} e^{-\langle z,u\rangle} P_t(x, du) = \exp\bigg\{ - x_1\psi_t(z_1) - x_2 z_2 - \int_0^t F(\psi_s(z_1) + z_2) ds\bigg\}, \tag{2.6}$$

where $x = (x_1, x_2) \in \mathbb{R}_+^2$, $z = (z_1, z_2) \in \mathbb{R}_+^2$ and $\psi_t(z_1)$ is the unique solution of

$$\frac{d\psi_t}{dt}(z_1) = R(\psi_t(z_1)), \qquad \psi_0(z_1) = z_1. \tag{2.7}$$

Remark 2.1 (i) The process $Y(\cdot)$ is a conservative *continuous state branching process with immigration* (CBI-process) and $Y'(\cdot)$ is the immigration part of $Y(\cdot)$. See Kawazu and Watanabe [13] for a complete characterization of the class of CBI-processes. Furthermore, $(Y(\cdot), Y'(\cdot))$ is a special case of two-dimensional CBI-processes; see [21] and the references therein.

(ii) Lemma 2.1-2.4 are closely related to the Lévy-Khinchin type representations of some class of continuous functions. Li [17] used a simple method based on Bernstein polynomials to prove Lemma 2.1 and 2.2. Here, inspired by Venttsel' [23], these lemmas can be proved in a different but more intuitive way. From the proof of Lemma 2.3, we can see that either (A) or (D2) implies the convergence of the sums of the triangular array of i.i.d. variables $\{\frac{\xi_n(k,j)-1}{b_n} : k, j = 1, 2\cdots\}$ or $\{\frac{\xi_n(k,j)-1}{c_n} : k, j = 1, 2\cdots\}$. See Grimvall [7] for a similar consideration.

Corollary 2.1 *([13]) Let $L(x)$ and $L^*(x)$ be positive functions slowly varying at ∞ such that $L(x) \sim L^*(x)$ as $x \to \infty$. Consider a sequence of GWI-processes with (g_n, h_n) given by*

$$\begin{aligned} g_n(s) &\equiv s + \gamma(1-s)^\alpha L\Big(\frac{1}{1-s}\Big), \\ h_n(s) &\equiv 1 - \varpi(1-s)^{\alpha-1} L^*\Big(\frac{1}{1-s}\Big), \end{aligned}$$

where $1 < \alpha \leq 2$, $\gamma > 0$, $\varpi > 0$. Let b_n be the sequence satisfying

$$b_n \sim [nL(b_n)]^{1/(\alpha-1)} \quad (\sim [nL^*(b_n)]^{1/(\alpha-1)}). \tag{2.8}$$

Then $(Y_n(\cdot)/b_n, Y_n'(\cdot)/b_n)$ converges in distribution on $D([0,\infty), \mathbb{R}_+^2)$ to $(Y(\cdot), Y'(\cdot))$ defined by

$$dY(t) = \sqrt[\alpha]{Y(t-)}\, dX(t) + dY'(t), \qquad Y(0) = Y'(0) = 0, \tag{2.9}$$

where $X(t)$ is a spectrally positive α-stable Lévy process with Laplace exponent $R(\lambda) = -\gamma\lambda^\alpha$, and $Y'(t)$ is a $(\alpha-1)$-stable subordinator with Laplace exponent $F(\lambda) = \varpi\lambda^{\alpha-1}$, independent of X.

Remark 2.2 Sometimes the above process can be regarded as the branching process conditioned on not being extinct in the distant future, or Q-process; see Lambert [16]. The pathwise uniqueness for the type of SDE (2.9) has recently been proved by Fu and Li [6].

Proof of Corollary 2.1 Without loss of generality, consider $R_n(\lambda)$ by (2.2) on $\lambda \in [0,1]$. It is easy to see that $\lim_{n\to\infty} R_n(\lambda) = -\gamma\lambda^\alpha$. For $\lambda > 0$, $|R_n'(\lambda)| = \alpha\gamma\lambda^{\alpha-1}[nL^\diamond(b_n/\lambda)/b_n^{\alpha-1}]$, where $L^\diamond(x)$ is a slowly varying functions such that $L^\diamond(x) \sim L(x)$, as $x \to \infty$ (cf. [2, Theorem 1.8.2]). By the representation theorem of $L^\diamond(\cdot)$ ([2, Theorem 1.3.1]),

$$L^\diamond(b_n/\lambda)/L^\diamond(b_n) = \{q(b_n/\lambda)/q(b_n)\}\exp\left\{\int_{b_n}^{b_n/\lambda} \epsilon(u)du/u\right\},$$

where $q(x) \to q \in (0,\infty)$, $\epsilon(x) \to 0$ as $x \to \infty$. Fix $0 < \varepsilon < q \wedge (\alpha - 1)$. There exists $x_0 > 0$ such that $q - \varepsilon < q(x) < q + \varepsilon$ and $|\epsilon(x)| < \varepsilon$, if $x > x_0$. Note that $b_n/\lambda \geq b_n$ for $0 < \lambda < 1$ and choose sufficiently large n, we have

$$\lambda^{\alpha-1}[L^\diamond(b_n/\lambda)/L^\diamond(b_n)] \leq \frac{q+\varepsilon}{q-\varepsilon}\lambda^{\alpha-1-\varepsilon}.$$

Then, by the above inequality and (2.8), $\sup_n |R_n'(\lambda)|$ is bounded in $\lambda \in (0,1]$. Note that $R_n'(0) = 0$ and thus (A) holds. Also, it is not hard to see that $\lim_{n\to\infty} F_n(\lambda) = \varpi\lambda^{\alpha-1}$. By Theorem 2.1, the limit process $(Y(\cdot), Y'(\cdot))$ is defined by (2.6) and (2.7) with $R(\lambda) = -\gamma\lambda^\alpha$ and $F(\lambda) = \varpi\lambda^{\alpha-1}$. By [6], $(Y(\cdot), Y'(\cdot))$ is the unique solution of the above stochastic equation system. $\square$

Now we turn to study the fluctuation limit for the sequence $Y_n(\cdot)$. Assume that $m_n = g_n'(1)$ and $\omega_n = h_n'(1)$ are finite. Let $\{c_n\}$ be a sequence of positive numbers. Set

$$G_n(\lambda) = n^2[(1 - m_n\lambda/c_n) - g_n(1 - \lambda/c_n)] \tag{2.10}$$

and

$$H_n(\lambda) = n[(1 - \omega_n\lambda/c_n) - h_n(1 - \lambda/c_n)], \tag{2.11}$$

for $0 \leq \lambda \leq c_n$. We will need the following conditions:

- (C) $n/c_n \to \infty$ and $n/c_n^2 \to \gamma_0$ as $n \to \infty$, for some $\gamma_0 \geq 0$;
- (D1) $n(m_n - 1) \to a$ as $n \to \infty$, for some $a \in \mathbb{R}$;
- (D2) The sequence $\{G_n\}$ is uniformly Lipschitz on each bounded interval and converges to a continuous function as $n \to \infty$;
- (E1) $\omega_n \to \omega$ as $n \to \infty$, for some $\omega \geq 0$;
- (E2) The sequence $\{H_n\}$ is uniformly Lipschitz on each bounded interval and converges to a continuous function as $n \to \infty$.

Lemma 2.3 *Under conditions (C) and (D1,2), the limit function G of $\{G_n\}$ has representation*

$$G(\lambda) = \beta_1\lambda - \sigma_1\lambda^2 - \int_0^\infty (e^{-\lambda u} - 1 + \lambda u)\mu(du), \tag{2.12}$$

where $\beta_1 \in \mathbb{R}$, $\sigma_1 \geq 0$ and $2\sigma_1 \geq a\gamma_0$, and $\mu(du)$ is a σ-finite measure on $(0,\infty)$ with $\int_0^\infty (u \wedge u^2)\mu(du) < \infty$.

Lemma 2.4 *Under conditions (C) and (E1,2), the limit function H of $\{H_n\}$ has representation*

$$H(\lambda) = \beta_2\lambda - \sigma_2\lambda^2 - \int_0^\infty (e^{-\lambda u} - 1 + \lambda u)\nu(du), \tag{2.13}$$

where $\beta_2 \in \mathbb{R}$, $\sigma_2 \geq 0$ and $2\sigma_2 + \omega\gamma_0 \geq \omega^2\gamma_0$, and $\nu(du)$ is a σ-finite measure on $(0,\infty)$ with $\int_0^\infty (u \wedge u^2)\nu(du) < \infty$.

Let $\phi(s) = \omega\int_0^s e^{au}du$ and let $\varrho(s) = (2\sigma_1 - a\gamma_0)\phi(s) + 2\sigma_2 + \omega(1-\omega)\gamma_0$. By the above representations, $\varrho(s) \geq 0$ for $s \geq 0$. We actually obtain a set of parameters $(\beta_1, \beta_2, \varrho(\cdot), \phi(\cdot), \mu, \nu)$ which will be used to characterizes our limit processes. Let $Z_n(\cdot)$ be defined by

$$Z_n(t) = \frac{Y_n(t) - \mathbf{E}[Y_n(t)]}{c_n}. \tag{2.14}$$

Our main result of this paper is the following fluctuation limit theorem.

Theorem 2.2 *Suppose that conditions (C), (D1,2) and (E1,2) are satisfied. Let $B(t)$ be a one-dimensional Brownian motion, $N_0(ds, du)$ be a Poisson random measure on $(0,\infty) \times \mathbb{R}_+$ with intensity $ds\nu(du)$ and $N_1(ds, du, d\zeta)$ be a Poisson random measure on $(0,\infty) \times \mathbb{R}_+ \times (0,\infty)$ with intensity $ds\mu(du)d\zeta$. Suppose that B, N_0 and N_1 are independent of each other. Then $Z_n(\cdot)$ converges in distribution on $D([0,\infty), \mathbb{R})$ to a time-inhomogeneous OU type process $Z(\cdot)$, which can be constructed as the unique solution of the following stochastic equation*

$$\begin{aligned} Z(t) = & \int_0^t \big(\beta_2 + \beta_1\phi(s) + aZ(s)\big)\, ds + \int_0^t \sqrt{\varrho(s)}\, dB(s) \\ & + \int_0^t \int_{\mathbb{R}_+} u\tilde{N}_0(ds, du) + \int_0^t \int_{\mathbb{R}_+} \int_0^{\phi(s)} u\tilde{N}_1(ds, du, d\zeta), \end{aligned} \tag{2.15}$$

where $\tilde{N}_0(ds, du) = N_0(ds, du) - ds\nu(du)$ and $\tilde{N}_1(ds, du, d\zeta) = N_1(ds, du, d\zeta) - ds\mu(du)d\zeta$.

Remark 2.3 The conditions of Theorem 2.2 imply that $Y_n(t)/n$ converges weakly to the deterministic function $\phi(t) = \omega\int_0^t e^{as}ds$. In fact, consider R_n in (2.2) and F_n in (2.3) with $b_n = n$. Note that $R_n(\lambda) = G_n(c_n\lambda/n) + n(m_n - 1)\lambda$. Then by conditions (C), (D1,2) and Lemma 2.1, $R_n'(\lambda)$ is uniformly bounded in each bounced interval and $\lim_{n\to\infty} R_n(\lambda) = a\lambda$. In a similar way, we also have $\lim_{n\to\infty} F_n(\lambda) = \omega\lambda$. The above weak convergence result follows from Theorem 2.1.

Corollary 2.2 *Let $\{y_n(k)\}$ be defined as in the beginning of this section. In addition to conditions (D1) and (E1), we assume that $\pi_n = \mathbf{var}\,\xi_n(1,1) < \infty$, $r_n = \mathbf{var}\,\eta_n(1) < \infty$, and the following conditions hold:*

(a.1) The sequence $\tilde{\mu}_n(\cdot) = n\mathbf{E}\Big[(\xi_n(1,1) - m_n)^2 \mathbf{1}_{\{(\xi_n(1,1)-m_n)/\sqrt{n}\, \in \cdot\}}\Big]$ converges weakly to a finite measure denoted by $\tilde{\mu}(\cdot)$, as $n \to \infty$;

(a.2) The sequence $\tilde{\nu}_n(\cdot) = \mathbf{E}\Big[(\eta_n(1) - \omega_n)^2 \mathbf{1}_{\{(\eta_n(1)-\omega_n)/\sqrt{n}\, \in \cdot\}}\Big]$ converges weakly to a finite measure denoted by $\tilde{\nu}(\cdot)$, as $n \to \infty$.

Let $Z_n(\cdot)$ be defined by (2.14) with $c_n = \sqrt{n}$. Then $Z_n(\cdot)$ converges in distribution on $D([0,\infty), \mathbb{R})$ to a OU type process $Z(\cdot)$ whose Lévy measure has finite second moment, i.e.

$$\begin{aligned} Z(t) = & \; a\int_0^t Z(s)\, ds + \int_0^t \sqrt{\varrho(s)}\, dB(s) + \int_0^t \int_{\mathbb{R}_+} u\tilde{N}_0(ds, du) \\ & + \int_0^t \int_{\mathbb{R}_+} \int_0^{\phi(s)} u\tilde{N}_1(ds, du, d\zeta), \end{aligned} \tag{2.16}$$

where $\varrho(s) = \tilde{\nu}(\{0\}) + \tilde{\mu}(\{0\})\phi(s)$ and similarly $\phi(s) = \omega \int_0^s e^{au} du$. B, N_0 and N_1 are defined as in (2.15), but the corresponding intensities of N_0 and N_1 are given by $ds\nu(du)$ and $ds\mu(du)d\zeta$ with $\nu(du) = u^{-2}\mathbf{1}_{\{u>0\}}\tilde{\nu}(du)$ and $\mu(du) = u^{-2}\mathbf{1}_{\{u>0\}}\tilde{\mu}(du)$.

Proof. By Theorem 2.2, it suffices to check (D2) and (E2) are satisfied. First it follows from (a.1,2) that $\tilde{\mu}(\cdot)$ and $\tilde{\nu}(\cdot)$ are supported by $[0,\infty)$. Consider $G_n(\lambda)$ in (2.10) with $c_n = \sqrt{n}$. Without loss of generality, we restrict ourselves to $\lambda \in [0,1]$. Making a Taylor expansion of g_n about 1, we have $G_n(\lambda) = -n\int_0^1 (1-s) g_n''(1 - s\lambda/\sqrt{n})\lambda^2 ds$. Note that

$$\begin{aligned} n g_n''(1-s\lambda/\sqrt{n}) \;=\; & (1-s\lambda/\sqrt{n})^{m_n-2}\Big(\int e^{u\sqrt{n}\ln(1-s\lambda/\sqrt{n})}\,\tilde{\mu}_n(du) \\ & + n m_n(m_n-1)\mathbf{E}[(1-s\lambda/\sqrt{n})^{\xi_n(1,1)-m_n}] \\ & + n(2m_n-1)\mathbf{E}[(\xi_n(1,1)-m_n)(1-s\lambda/\sqrt{n})^{\xi_n(1,1)-m_n}]\Big). \end{aligned}$$

Fix $s,\lambda \in (0,1]$ and choose sufficiently large n such that $0 < s\lambda/\sqrt{n} \le 1/2$. We have that

$$\Big|\int e^{u\sqrt{n}\ln(1-s\lambda/\sqrt{n})}\,\tilde{\mu}_n(du) - \int e^{-us\lambda}\tilde{\mu}_n(du)\Big| \le \int (e^{-us\lambda}|u|/\sqrt{n}) \vee (e(e^{1/\sqrt{n}}-1))\tilde{\mu}_n(du),$$

$$\big|1 - (1-s\lambda/\sqrt{n})^{\xi_n(1,1)-m_n}\big| \le 2e^2|\xi_n(1,1)-m_n|/\sqrt{n}.$$

Note that $\tilde{\mu}_n(\cdot)$ is supported by $\{(k-m_n)/\sqrt{n} : k = 0,1,\cdots\}$. By conditions (D1), (E1) and (a.1), it is easy to see that $ng''(1-s\lambda/\sqrt{n}) \to a + \int_{[0,\infty)} e^{-us\lambda}\tilde{\mu}(du)$. Since $ng''(1-s\lambda/\sqrt{n})$ is bounded, (D2) holds and $\lim_{n\to\infty} G_n(\lambda) = -(\tilde{\mu}(\{0\}) + a)\lambda^2/2 - \int_{(0,\infty)}(e^{-\lambda u} - 1 + \lambda u)/u^2\tilde{\mu}(du)$. It follows in a similar way that (E2) also holds and $\lim_{n\to\infty} H_n(\lambda) = -(\tilde{\nu}(\{0\}) + \omega^2 - \omega)\lambda^2/2 - \int_{(0,\infty)}(e^{-\lambda u} - 1 + \lambda u)/u^2\tilde{\nu}(du)$. □

Example 2.1 ([11, Theorem 2.2]) Assume that (D1), (E1) and the following conditions hold:

(b.1) $n\pi_n \to \pi$ and $r_n \to r$ as $n \to \infty$ for some $\pi \ge 0$ and $r \ge 0$,

(b.2) $n\mathbf{E}\Big[(\xi_n(1,1)-m_n)^2\mathbf{1}_{\{|\xi_n(1,1)-m_n|>\sqrt{n}\,\varepsilon}\Big] \to 0$ as $n\to\infty$ for all $\varepsilon > 0$,

(b.3) $\mathbf{E}\Big[(\eta_n(1)-\omega_n)^2\mathbf{1}_{\{|\eta_n(1)-\omega_n|>\sqrt{n}\,\varepsilon\}}\Big] \to 0$ as $n\to\infty$ for all $\varepsilon > 0$.

In this case, we see that conditions (a.1,2) are satisfied with $\tilde{\mu}(du) = \pi\delta_0(du)$ and $\tilde{\nu}(du) = r\delta_0(du)$ ($\delta_x(du)$ denote the dirac measure at $u = x$). Then we still have the above limit theorem, and the fluctuation limit process $Z(\cdot)$ is given by

$$dZ(t) = aZ(t)dt + \sqrt{\varrho(t)}\,dB(t), \qquad Z(0) = 0, \tag{2.17}$$

where $\varrho(t) = r + \pi\phi(t)$, and $B(t)$ is a one-dimensional Brownian motion.

Example 2.2 Suppose that $\mathbf{P}(\xi_n(1,1) = [\sqrt{n}]) = 1/n^2$ and $\mathbf{P}(\xi_n(1,1) = 1) = 1 - 1/n^2$, while $\mathbf{P}(\eta_n(1) = [\sqrt{n}]) = 1/n$ and $\mathbf{P}(\eta_n(1) = 1) = 1 - 1/n$. We see that $n(m_n - 1) \to 0$, $\omega_n \to 1$ and (a.1,2) are satisfied with $\tilde{\mu}(du) = \tilde{\nu}(du) = \delta_1(du)$. Another example is as follows. Suppose that $\tilde{\mu}(du)$ is any non-degenerate finite measure on $(0,\infty)$. For large enough n, let $\mu(du) = u^{-2}\tilde{\mu}(du)$, $\mu_n(du) = \mu((1/n^{1/4},\infty))^{-1}1_{\{u>n^{1/4}\}}\mu(du)$, and

$$g_n(s) = p_n\int_0^\infty e^{-\sqrt{n}u(1-s)}\mu_n(du) + (1-p_n), \quad \text{where} \quad p_n = \frac{\mu((1/n^{1/4},\infty))}{\mu((1/n^{1/4},\infty)) + n^2}.$$

Let $\xi_n(1,1)$ have the distribution corresponding to $g_n(\cdot)$. Note that $\int_{\{u>1/n^{1/4}\}} u\mu(du)/\sqrt{n} \to 0$ and $\mu((1/n^{1/4},\infty))/n \to 0$. Then it is not hard to see that $n(m_n-1)\to 0$ and condition (a.1) is fulfilled with $\tilde{\mu}(du)$. $\eta_n(1)$ can be constructed in a similar way.

Corollary 2.3 *Let $L(x)$ and $L^*(x)$ be positive functions slowly varying at ∞ such that $L(x)\sim L^*(x)$ as $x\to\infty$. Consider a sequence of GWI-processes with (g_n,h_n) given by*

$$\begin{aligned} g_n(s) &= (1-m_n)+m_n s+\frac{\gamma}{n}(1-s)^\alpha L\Big(\frac{1}{1-s}\Big), \\ h_n(s) &= (1-\omega_n)+\omega_n s+\varpi(1-s)^\alpha L^*\Big(\frac{1}{1-s}\Big), \end{aligned}$$

where $1<\alpha\le 2$, $\gamma>0$, $\varpi>0$. m_n and ω_n satisfy conditions (D1) and (E1). Let $Z_n(\cdot)$ be defined by (2.14) with c_n satisfying

$$c_n \sim [nL(c_n)]^{1/\alpha} \quad (\sim [nL^*(c_n)]^{1/\alpha}). \tag{2.18}$$

Then $Z_n(\cdot)$ converges in distribution on $D([0,\infty),\mathbb{R})$ to a OU type process $Z(\cdot)$ defined by

$$dZ(t) = aZ(t)dt + \sqrt[\alpha]{\varrho_1(t)}\, dX(t), \qquad Z(0)=0, \tag{2.19}$$

where $\varrho_1(t)=\varpi+\gamma\phi(t)$, $\phi(t)=\omega\int_0^t e^{au}du$, and $X(t)$ is a spectrally positive α-stable Lévy process with Laplace exponent $-\lambda^\alpha$.

Proof. By (2.18), $c_n/n \sim c_n^{1-\alpha}L(c_n)\to 0$ and $c_n^2/n\sim c_n^{2-\alpha}L(c_n)\to\infty$, as $n\to\infty$. Then (C) holds. Without loss of generality, we consider $G_n(\lambda)$ on $\lambda\in[0,1]$ (see (2.10)). For $\lambda>0$, $G_n(\lambda)=-\gamma\lambda^\alpha[nL(c_n)/c_n^\alpha][L(c_n/\lambda)/L(c_n)]$, and thus $\lim_{n\to\infty}G_n(\lambda)=-\gamma\lambda^\alpha$. For $\lambda=0$, the limit is trivial. Furthermore we have $\sup_n|G_n'(\lambda)|$ is bounded in $\lambda\in(0,1]$ as in the proof of Corollary 2.1. Note that $G_n'(0)=0$ and thus (D2) holds. It follows in a similar way that (E2) also holds and $\lim_{n\to\infty}H_n(\lambda)=-\varpi\lambda^\alpha$. By Theorem 2.2, the limit process $Z(\cdot)$ is described by (2.15) with $\mu(du)=(\gamma\alpha(\alpha-1)/\Gamma(2-\alpha))u^{-1-\alpha}du$, $\nu(du)=(\varpi\alpha(\alpha-1)/\Gamma(2-\alpha))u^{-1-\alpha}du$, $\varrho(\cdot)\equiv 0$ and $\beta_1=\beta_2=0$. Note that $\varrho_1(t)>0$ for $t\ge 0$. Define the process

$$X(t)=\int_0^t\int_{\mathbb{R}_+}\varrho_1(s)^{-\frac{1}{\alpha}}u\tilde{N}_0(ds,du)+\int_0^t\int_{\mathbb{R}_+}\int_0^\infty\varrho_1(s)^{-\frac{1}{\alpha}}u\mathbf{1}_{(0,\phi(s)]}(\zeta)\tilde{N}_1(ds,du,d\zeta). \tag{2.20}$$

Then $X(\cdot)$ is a martingale. By Itô's formula, it is not hard to show that $X(\cdot)$ is a one-sided α-stable process with Laplace exponent $-\lambda^\alpha$. Thus we have (2.19) by (2.20) and (2.15). □

3 Asymptotic results for estimators

In this section, we consider the statistical applications of our limit theorems as in [22, 11]. For $n\in\mathbb{N}$, suppose that a sequence of samples $\{(y_n(k),\eta_n(k)), k=1,2,\cdots,n\}$ is available. Then a natural estimator of the offspring mean m_n is given by

$$\check{m}_n=\Big[\sum_{k=1}^n y_n(k-1)\Big]^{-1}\sum_{k=1}^n(y_n(k)-\eta_n(k)).$$

Using Theorem 2.1, we can derive the following asymptotic result for $\check{m}_n$.

Theorem 3.1 *If the conditions of Theorem 2.1 are fulfilled with $F(\lambda)$ being a unbounded function, and $R_n'(0) \to a$ as $n \to \infty$ for some $a \in \mathbb{R}$, then*

$$n(\check{m}_n - m_n) \xrightarrow{d} \frac{Y(1) - a\int_0^1 Y(s)ds - Y'(1)}{\int_0^1 Y(t)dt}, \tag{3.1}$$

where $Y(t)$ and $Y'(t)$ are defined in Theorem 2.1.

Obviously, the above theorem applies to the case of Corollary 2.1. Compared with the result of [22, Corollary 3.1], this implies that the heavy-tailed stable distributions of offspring and immigration variables do not affect the rate of convergence of $\check{m}_n$ in the critical GWI-process.

We are also interested in the case when the conditions of Theorem 2.2 are satisfied. Then $Y_n(\cdot)/n$ converges weakly to the deterministic function $\phi(t) = \omega\int_0^t e^{as}ds$, which implies only that $n(\check{m}_n - m_n) \xrightarrow{p} 0$ by Theorem 3.1. Thus we further consider the applications of our fluctuation limit theorem, related to the CLSE of the offspring mean m_n based on only the information on $\{y_n(k)\}$ as follows. For $n, k \geq 1$ let $\mathcal{F}_k^n$ denote the σ-algebra generated by $\{y_n(j), j = 0, 1, \cdots, k\}$. From (2.1),

$$\mathbf{E}[y_n(k)|\mathcal{F}_{k-1}^n] = m_n y_n(k-1) + \omega_n, \qquad n \geq 1. \tag{3.2}$$

If we assume that the immigration mean ω_n is known, then the CLSE $\hat{m}_n$ of m_n, based on (3.2), is given by

$$\hat{m}_n = \frac{\sum_{k=1}^n y_n(k-1)(y_n(k) - \omega_n)}{\sum_{k=1}^n y_n^2(k-1)}. \tag{3.3}$$

If ω_n is unknown, it is not hard to see that the joint CLSE $(\tilde{m}_n, \tilde{\omega}_n)$ of (m_n, ω_n) is given by

$$\tilde{m}_n = \frac{\sum_{k=1}^n y_n(k-1)(y_n(k) - \overline{y_n})}{\sum_{k=1}^n (y_n(k-1) - \overline{y_n^*})^2}, \qquad \tilde{\omega}_n = \overline{y_n} - \tilde{m}_n\overline{y_n^*},$$

where

$$\overline{y_n} = \frac{1}{n}\sum_{k=1}^n y_n(k), \qquad \overline{y_n^*} = \frac{1}{n}\sum_{k=1}^n y_n(k-1). \tag{3.4}$$

Using Theorem 2.2, we can derive the following asymptotic result for $\hat{m}_n$, $\tilde{m}_n$, and $\tilde{\omega}_n$, which generalizes the result of [11, Theorem 3.1].

Theorem 3.2 *If the conditions of Theorem 2.2 are fulfilled with $\omega > 0$, then*

$$\frac{n^2}{c_n}(\hat{m}_n - m_n) \xrightarrow{d} \frac{\int_0^1 \phi(t)dM(t)}{\int_0^1 \phi^2(t)dt}, \tag{3.5}$$

where $\phi(t) = \omega\int_0^t e^{as}ds$, $Z(t)$ is defined by (2.15), $M(t) = Z(t) - \int_0^t aZ(s)ds$ and it can be regarded as a deterministically-time-changed Lévy process. Furthermore,

$$\begin{pmatrix} \frac{n^2}{c_n}(\tilde{m}_n - m_n) \\ \\ \frac{n}{c_n}(\tilde{\omega}_n - \omega_n) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \frac{\int_0^1 \phi(t)dM(t) - M(1)\int_0^1 \phi(t)dt}{\int_0^1 \phi^2(t)dt - (\int_0^1 \phi(t)dt)^2} \\ \\ \frac{M(1)\int_0^1 \phi^2(t)dt - \int_0^1 \phi(t)dt\int_0^1 \phi(t)dM(t)}{\int_0^1 \phi^2(t)dt - (\int_0^1 \phi(t)dt)^2} \end{pmatrix}. \tag{3.6}$$

Corollary 3.1 *Consider the CLSE $\hat{m}_n$ of m_n in the case of Corollary 2.3 when condition (E1) holds with $\omega > 0$. By the above theorem, we have*

$$\frac{n^2}{c_n}(\hat{m}_n - m_n) \xrightarrow{d} U := \frac{\int_0^1 \phi(t) \sqrt[\alpha]{\varrho_1(t)} dX(t)}{\int_0^1 \phi^2(t)dt},$$

where $1 < \alpha \le 2$, c_n is given by (2.18), $\phi(t)$, $\varrho_1(t)$ and $X(t)$ are given in (2.19). It is easy to see that U has a α-stable distribution and its Laplace transform equals

$$\mathbf{E}[e^{-\lambda U}] = \exp\left\{ \frac{\int_0^1 \phi^\alpha(t)\varrho_1(t)dt}{(\int_0^1 \phi^2(t)dt)^\alpha} \lambda^\alpha \right\}, \qquad \lambda \ge 0.$$

Finally we turn to the case when the conditions of Corollary 2.2 are satisfied. In this case, it is possible to consider the CLSE estimates for the offspring and immigration variances π_n and r_n. Let $u_n(k) = y_n(k) - m_n y_n(k-1) - \omega_n$. Note that

$$\mathbf{E}[u_n^2(k)|\mathcal{F}_{k-1}^n] = \pi_n y_n(k-1) + r_n, \qquad n \ge 1. \tag{3.7}$$

As in [26], if we suppose that m_n and ω_n are known, then the joint CLSE $(\hat{\pi}_n, \hat{r}_n)$ of (π_n, r_n), based on (3.7), is given by

$$\hat{\pi}_n = \frac{\sum_{k=1}^n u_n^2(k)(y_n(k-1) - \overline{y_n^*})}{\sum_{k=1}^n (y_n(k-1) - \overline{y_n^*})^2}, \qquad \hat{r}_n = \sum_{k=1}^n u_n^2(k)/n - \hat{\pi}_n \overline{y_n^*}, \tag{3.8}$$

where $\overline{y_n^*}$ is defined by (3.3). If m_n and ω_n are unknown, we can use $\hat{u}_n(k) = y_n(k) - \hat{m}_n y_n(k-1) - \hat{\omega}_n$ instead of $u_n(k)$ in (3.8) and we get another joint CLSE denoted by $(\tilde{\pi}_n, \tilde{r}_n)$. Using Theorem 2.2 again, we have the following asymptotic result for the above estimators, where the jumps of the fluctuation limit obviously play an important role.

Theorem 3.3 *If the conditions of Corollary 2.2, i.e. (D1), (E1) and (a.1,2) are fulfilled with $\omega > 0$, then*

$$\begin{pmatrix} n(\hat{\pi}_n - \pi_n) \\ \\ \hat{r}_n - r_n \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \frac{\int_0^1 \phi(t)dJ(t) - J(1)\int_0^1 \phi(t)dt}{\int_0^1 \phi^2(t)dt - (\int_0^1 \phi(t)dt)^2} \\ \\ \frac{J(1)\int_0^1 \phi^2(t)dt - \int_0^1 \phi(t)dt \int_0^1 \phi(t)dJ(t)}{\int_0^1 \phi^2(t)dt - (\int_0^1 \phi(t)dt)^2} \end{pmatrix}, \tag{3.9}$$

where $\phi(t) = \omega \int_0^t e^{as}ds$, and $J(t)$ is a martingale defined by

$$J(t) = \int_0^t \int_{\mathbb{R}_+} u^2 \tilde{N}_0(ds, du) + \int_0^t \int_{\mathbb{R}_+} \int_0^{\phi(s)} u^2 \tilde{N}_1(ds, du, d\zeta). \tag{3.10}$$

Here $\tilde{N}_0$ and $\tilde{N}_1$ are the compensated Poisson random measures given in (2.16). Moreover, (3.9) still holds if $\hat{\pi}_n$ and $\hat{r}_n$ are replaced by $\tilde{\pi}_n$ and $\tilde{r}_n$.

Remark 3.1 N_0 and N_1 are the Poisson random measures given in (2.16) with intensities $ds\nu(du)$ and $ds\mu(du)d\zeta$, and $\int_0^\infty u^2\nu(du) + \int_0^\infty u^2\mu(du) < \infty$. Let the limiting random vector in (3.9) be denoted by $(U_1, U_2)^T$. It is not hard to see that if $\int_0^\infty u^4\nu(du) + \int_0^\infty u^4\mu(du) < \infty$, then

$$\begin{aligned} \mathbf{E}[U_1^2] &= L^{-2}\Big[\int_0^1 \Big(\phi(t) - \int_0^1 \phi(s)ds\Big)^2 dt \int_0^\infty u^4\nu(du) \\ &\qquad + \int_0^1 \phi(t)\Big(\phi(t) - \int_0^1 \phi(s)ds\Big)^2 dt \int_0^\infty u^4\mu(du)\Big], \end{aligned}$$

where $L = \int_0^1 \phi^2(t)dt - (\int_0^1 \phi(t)dt)^2$. Otherwise we have $\mathbf{E}[U_1^2] = \infty$. Note that $\phi(\cdot)$ is not a const function. So if $\nu \neq 0$ or $\mu \neq 0$ (equivalently N_0 or N_1 is not degenerate), then U_1, and similarly U_2, are not degenerate.

We see that if the conditions of Corollary 2.2 are fulfilled with $\tilde{\nu}(\mathbb{R}_+\backslash\{0\}) > 0$ or $\tilde{\mu}(\mathbb{R}_+\backslash\{0\}) > 0$, which means that the sequence of the offspring (or immigration) distributions fails to satisfy Lindeberg condition (b.2) (or (b.3)), then the resulting fluctuation limit $Z(\cdot)$ is a OU type process with positive jumps (see (2.16)). Thus, in this case, $\hat{\pi}_n$ has the limit law U_1 with normalizing factor n, and $\hat{r}_n$ is not a consistent estimator. However, if we return to the case of [11] (see Example 2.1), which implies that Lindeberg conditions are satisfied and the resulting fluctuation limit $Z(\cdot)$ is a OU diffusion process without jumps (see (2.17)), then $n(\hat{\pi}_n - \pi_n) \xrightarrow{p} 0$ and $\hat{r}_n - r_n \xrightarrow{p} 0$. In this case, to get the appropriate rates of convergence for $\hat{\pi}_n$ and $\hat{r}_n$, we give the following theorem.

Theorem 3.4 *Consider the case of Example 2.1. Let* $a_{4,n} = \mathbf{E}[(\xi_n(1,1) - m_n)^4]$ *and* $b_{4,n} = \mathbf{E}[(\eta_n(1) - \omega_n)^4]$. *Suppose that (D1), (E1), (b.1) and the following conditions hold with* $\omega > 0$:

(c.1) $na_{4,n} \to a_4$ *and* $b_{4,n} \to b_4$ *as* $n \to \infty$ *for some* $a_4 \geq 0$ *and* $b_4 \geq 0$,

(c.2) $n\mathbf{E}\Big[(\xi_n(1,1) - m_n)^4 \mathbf{1}_{\{(\xi_n(1,1)-m_n)^2 > \sqrt{n}\,\varepsilon\}}\Big] \to 0$ *as* $n \to \infty$ *for all* $\varepsilon > 0$,

(c.3) $\mathbf{E}\Big[(\eta_n(1) - \omega_n)^4 \mathbf{1}_{\{(\eta_n(1)-\omega_n)^2 > \sqrt{n}\,\varepsilon\}}\Big] \to 0$ *as* $n \to \infty$ *for all* $\varepsilon > 0$.

Let $\phi(t) = \omega \int_0^t e^{as} ds$, $\varrho_2(t) = 2\pi^2\phi^2(t) + (a_4 + 4\pi r)\phi(t) + (b_4 - r^2)$ *and* $V(t) = \int_0^t \sqrt{\varrho_2(s)}\, dW(s)$, *where* $W(t)$ *is a one-dimensional Brownian motion. Let* $\Sigma = \big(\int_0^1 \phi^2(t)dt - (\int_0^1 \phi(t)dt)^2\big)^{-2} (\sigma_{ij})_{2\times 2}$, *where*

$$\sigma_{11} = \int_0^1 \big(\phi(t) - \int_0^1 \phi(s)ds\big)^2 \varrho_2(t)dt, \quad \sigma_{22} = \int_0^1 \big(\int_0^1 \phi^2(s)ds - \phi(t)\int_0^1 \phi(s)ds\big)^2 \varrho_2(t)dt,$$

$$\sigma_{12} = \sigma_{21} = \int_0^1 \big(\phi(t) - \int_0^1 \phi(s)ds\big)\big(\int_0^1 \phi^2(s)ds - \phi(t)\int_0^1 \phi(s)ds\big)\varrho_2(t)dt.$$

Then we have

$$\begin{pmatrix} n^{3/2}(\hat{\pi}_n - \pi_n) \\ \\ n^{1/2}(\hat{r}_n - r_n) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \frac{\int_0^1 \phi(t)dV(t) - V(1)\int_0^1 \phi(t)dt}{\int_0^1 \phi^2(t)dt - (\int_0^1 \phi(t)dt)^2} \\ \\ \frac{V(1)\int_0^1 \phi^2(t)dt - \int_0^1 \phi(t)dt \int_0^1 \phi(t)dV(t)}{\int_0^1 \phi^2(t)dt - (\int_0^1 \phi(t)dt)^2} \end{pmatrix} \stackrel{d}{=} \mathcal{N}(0, \Sigma). \tag{3.11}$$

Furthermore, (3.11) still holds if $\hat{\pi}_n$ *and* $\hat{r}_n$ *are replaced by* $\tilde{\pi}_n$ *and* $\tilde{r}_n$, *respectively.*

Remark 3.2 It is easy to see that the above condition (c.1) implies that (b.2) and (b.3) in Example 2.1 hold. So the conditions of our theorem is in the case of Example 2.1. If either of π, a_4 and $b_4 - r^2$ is not 0, then the limit normal law in (3.11) is not degenerate.

Example 3.1 ([11, Example 2.1]) The conditions of Theorem 3.4 are satisfied for the following examples with $n(m_n - 1) \to a$, $n\pi_n \to a$, $na_{4,n} \to a$ and $n\mathbf{E}[(\xi_n(1,1) - m_n)^6] \to a$, as $n \to \infty$ for some $a \geq 0$. (i) $\xi_n(1,1)$ has a Bernoulli distribution with mean $1 - an^{-1}$. (ii) the offspring distributions are geometric distributions with parameter $p_n = 1 - an^{-1}$, i.e. $\mathbf{P}(\xi_n(1,1) = i) = p_n(1 - p_n)^{i-1}$, $i = 1, 2, \cdots$.

4 Proof of main results

Proof of Theorem 2.1 For the proofs of Lemma 2.1 and 2.2, we can follow the proof of Lemma 2.3 or apply directly [17, Corollary 1,2]. So we skip them. Now the limit functions R and F have representations (2.4) and (2.5). Fix $0 \le \lambda \le M$ for any constant $M > 0$. Let $\lambda_n = b_n(1 - e^{-\lambda/b_n})$ and we have $\lambda_n \to \lambda$. It follows from condition (A) that $|R_n(\lambda_n) - R_n(\lambda)| \le k(M)|\lambda_n - \lambda|$, where $k(M) > 0$ is a constant, and that $\lim_{n\to\infty} R_n(\lambda_n) = R(\lambda)$. By condition (B) and the fact that F_n is a nondecreasing function on $\lambda \in [0, M]$ for sufficiently large n, we have $F_n \to F$ locally uniformly. It implies that $\lim_{n\to\infty} F_n(\lambda_n) = F(\lambda)$. Let $\tilde{R}_n(\lambda) = R_n(\lambda_n)$ and $\tilde{F}_n(\lambda) = F_n(\lambda)$. Note that the sequence $\{(Y_n(\frac{l}{n}), Y_n'(\frac{l}{n})), l \in \mathbb{N}\}$ is a Markov chain with state space $\hat{E}_n := \{(i/b_n, j/b_n) : (i,j) \in \mathbb{N}^2\}$ and the (discrete) generator A_n of $\{(Y_n(t), Y_n'(t)), t \ge 0\}$ is given by

$$\begin{aligned}
A_n e^{-\langle z,x\rangle} &= n\Big[\big(g_n(e^{-z_1/b_n})\big)^{b_n x_1} h_n(e^{-(z_1+z_2)/b_n}) e^{-z_2 x_2} - e^{-\langle z,x\rangle}\Big] \\
&= e^{-\langle z,x\rangle} n\Big[\exp\big\{-x_1\alpha_n(z)e^{z_1/b_n}\tilde{R}_n(z_1)/n\big\}\exp\big\{-\beta_n(z)\tilde{F}_n(z_1+z_2)/n\big\} - 1\Big] \\
&= -e^{-\langle z,x\rangle}\big[x_1\alpha_n(z)e^{z_1/b_n}\tilde{R}_n(z_1) + \beta_n(z)\tilde{F}_n(z_1+z_2)\big] + o(1),
\end{aligned}$$

where $x \in \hat{E}_n$, $z = (z_1, z_2) \gg 0$, $\alpha_n(z) = \big(e^{z_1/b_n} g_n(e^{-z_1/b_n}) - 1\big)^{-1} \ln\big(e^{z_1/b_n} g_n(e^{-z_1/b_n})\big)$, and $\beta_n(z) = \big(h_n(e^{-(z_1+z_2)/b_n}) - 1\big)^{-1} \ln\big(h_n(e^{-(z_1+z_2)/b_n})\big)$. On the other hand, let A be the infinitesimal generator of $(Y(\cdot), Y'(\cdot))$. For $z \gg 0$ and $x \in \mathbb{R}_+^2$,

$$A e^{-\langle z,x\rangle} = -e^{-\langle z,x\rangle}\big[x_1 R(z_1) + F(z_1+z_2)\big].$$

We need to prove that $\lim_{n\to\infty} \sup_{x\in\hat{E}_n} |A_n e^{-\langle z,x\rangle} - A e^{-\langle z,x\rangle}| = 0$. The remaining proof is essentially the same as that of in [19, Theorem 2.1] or [21, Theorem 2.1] and so we omit it. $\square$

Let us write $f \in C_*(\mathbb{R})$ if f is a bounded continuous function from $\mathbb{R}$ to $\mathbb{R}$ satisfying $f(x) = o(x^2)$ when $x \to 0$. Let $\Gamma = [-1, \infty)$ and $\Gamma_n = \{(i-1)/c_n : i \in \mathbb{N}\}$. Let μ_n be the distribution of $\frac{\xi_n(1,1)-1}{c_n}$. Then for sufficiently large n, μ_n is a probability measure on Γ supported by Γ_n.

Proof of Lemma 2.3 (sketch) Set $S_n(\lambda) = n^2\big[e^{-\lambda/c_n}\big(1 - (m_n - 1)\lambda/c_n\big) - g_n\big(e^{-\lambda/c_n}\big)\big]$ and it follows from mean-value theorem that

$$\begin{aligned}
S_n(\lambda) = &\ G_n(\lambda) + n^2[m_n - g_n'(\vartheta_n)](e^{-\lambda/c_n} - 1 + \lambda/c_n) \\
&+ n^2(1 - m_n)(e^{-\lambda/c_n} - 1 + \lambda/c_n) + n^2(m_n - 1)(1 - e^{-\lambda/c_n})\lambda/c_n,
\end{aligned} \tag{4.1}$$

where $1 - \lambda/c_n \le \vartheta_n \le e^{-\lambda/c_n}$. Under condition (D2), the sequence $|G_n'(\lambda)| = n^2|g_n'(1 - \lambda/c_n) - m_n|/c_n$ is uniformly bounded on each bounded interval $[0, c]$ for $c \ge 0$ and thus the sequence $n^2|g_n'(\vartheta_n) - m_n|/c_n$ is also uniformly bounded. By (C), (D1) and (D2), we have $S_n(\lambda) \to G(\lambda) + \frac{1}{2}a\gamma_0$, as $n \to \infty$. To get (2.12), it is enough to consider the limit representation of S_n. Note that

$$e^{\lambda/c_n} S_n(\lambda) = -n^2 \int_\Gamma (e^{-\lambda u} - 1 + \lambda u)\mu_n(du).$$

We can use Venttsel's classical method (see [23]) to prove it. More precisely, by modifying slightly the proofs of Proposition 2.1 and 3.1 in [20], we can show that there exist some constants $\hat{\beta}_1 \in \mathbb{R}$, $\hat{\sigma}_1 \ge 0$, and a σ-finite measure μ defined as in (2.12) such that

(i) $n^2 \displaystyle\int_\Gamma (\chi(u) - u)\mu_n(du) \to \hat{\beta}_1 \quad$ as $n \to \infty$;

(ii) $n^2 \int_\Gamma \chi^2(u)\mu_n(du) \to 2\hat{\sigma}_1 + \int_0^\infty \chi^2(u)\mu(du) \quad \text{as } n \to \infty;$

(iii) $\lim_{n\to\infty} n^2 \int_\Gamma f(u)\mu_n(du) = \int_0^\infty f(u)\mu(du), \quad \text{for } f \in C_*(\mathbb{R}).$

Note that $e^{-\lambda x} - 1 + \lambda\chi(x) - \frac{1}{2}\lambda^2\chi^2(x) \in C_*(\Gamma)$ as a function of $x \in \Gamma$ for fixed $\lambda \geq 0$. The above results imply that the limit function of S_n has a Lévy-Khintchine type representation. Let $\sigma_1 = \hat{\sigma}_1 + \frac{1}{2}a\gamma_0$ and let $\beta_1 = \hat{\beta}_1 + \int_0^\infty (u - \chi(u))\mu(du)$. Then we have (2.12). But we still need to verify $\sigma_1 \geq 0$. It follows from (a.1), (C) and (D1) that

$$\frac{n^2}{c_n}\int_\Gamma \chi(u)\mu_n(du) = \frac{n^2}{c_n}\int_\Gamma (\chi(u) - u)\mu_n(du) + \frac{n^2(m_n - 1)}{c_n^2}, \tag{4.2}$$

which tends to $a\gamma_0$ as $n \to \infty$. Let E be the set of $\varepsilon > 0$ for which $\mu(|u| = \varepsilon) = 0$. By (4.2), (ii) and (iii), we obtain

$$\lim_{E\ni\varepsilon\downarrow 0}\lim_{n\to\infty} n^2 \int_{\{|u|<\varepsilon\}} \Big(\chi^2(u) + \frac{\chi(u)}{c_n}\Big)\mu_n(du) = 2\sigma_1.$$

The support of μ_n is Γ_n and for large enough n, $\chi^2(u) + \big(\chi(u)/c_n\big) \geq 0$ if $u \in \Gamma_n$. Thus $\sigma_1 \geq 0$.□

Let $\hat{\Gamma}_n = \{i/c_n : i \in \mathbb{N}\}$ and let ν_n be the distribution of $\frac{\eta_n(1)}{c_n}$. Then ν_n is a probability measure on $[0, \infty)$ supported by $\hat{\Gamma}_n$.

Lemma 4.1 *Under conditions (C), (E1) and (E2), (2.13) holds. As $n \to \infty$, we also have*

(i) $n \int_0^\infty (\chi(u) - u)\nu_n(du) \to \beta_2 - \int_0^\infty (u - \chi(u))\nu(du);$

(ii) $n \int_0^\infty \chi^2(u)\nu_n(du) \to 2\sigma_2 + \omega\gamma_0 + \int_0^\infty \chi^2(u)\nu(du);$

(iii) $\lim_{n\to\infty} n \int_0^\infty f(u)\nu_n(du) = \int_0^\infty f(u)\nu(du), \quad \textit{for } f \in C_*(\mathbb{R}_+).$

Proof. This lemma is proved with the same method as Lemma 2.3. But we need to prove that $2\sigma_2 + \omega\gamma_0 \geq \omega^2\gamma_0$. Let $\hat{a}_n = \int_0^\infty \chi(u)\nu_n(du)$ and let $\hat{E}$ be the set of $\varepsilon > 0$ for which $\nu(u = \varepsilon) = 0$. By (C), (E1), (i) and (ii), it is not hard to show that

$$\lim_{\hat{E}\ni\varepsilon\downarrow 0}\lim_{n\to\infty} n \int_{\{u<\varepsilon\}} \big(\chi^2(u) - \frac{\chi(u)}{c_n}\big)\nu_n(du) = 2\sigma_2,$$
$$\lim_{\hat{E}\ni\varepsilon\downarrow 0}\lim_{n\to\infty} n \int_{\{u<\varepsilon\}} (\chi(u) - \hat{a}_n)^2\nu_n(du) = 2\sigma_2 + \omega\gamma_0 - \omega^2\gamma_0.$$

For large enough n, $\chi^2(u) - \big(\chi(u)/c_n\big) \geq 0$ if $u \in \hat{\Gamma}_n$. Then we are finished.

Lemma 4.2 *Under the conditions of Theorem 2.2, we have for $t \geq 0$,*

$$\limsup_{n\to\infty} \frac{1}{n}\mathbf{E}\Big[\sup_{0\leq s\leq t} Y_n(s)\Big] \leq |a|\Phi(t) + \omega t, \tag{4.3}$$

$$\limsup_{n\to\infty} \mathbf{E}\Big[\sup_{0\le s\le t} |Z_n(s)|\Big] \le \hat{M}(t)\big[1+(|a|+1)t\,\exp\{(|a|+1)t\}\big], \tag{4.4}$$

where $\Phi(t) = \int_0^t \int_0^s e^{(|a|+1)u} duds$, $\hat{M}(t) = 2K(\Phi(t)+t) + 4\sqrt{K(\Phi(t)+t)}$, *and* K *is a positive constant defined as in (4.9).*

Proof. Note that (2.1) can be rewritten into the following form:

$$y_n(l) = \sum_{k=1}^{l} \sum_{j=1}^{y_n(k-1)} (\xi_n(k,j)-1) + \sum_{k=1}^{l} \eta_n(k). \tag{4.5}$$

Let $\hat{\xi}_n(k,j) = (\xi_n(k,j)-1)/c_n$, $w_n(k) = \sum_{j=1}^{y_n(k-1)} \big(\chi(\hat{\xi}_n(k,j)) - \mathbf{E}[\chi(\hat{\xi}_n(k,j))]\big)$, and $W_n(l) = \sum_{k=1}^{l} w_n(k)$. let $\tilde{\mathcal{F}}_k^n$ denote the σ-algebra generated by $\{(w_n(j), y_n(j)), j = 0,1,\cdots,k\}$. Since $\mathbf{E}[w_n(k)|\tilde{\mathcal{F}}_{k-1}^n] = 0$, $W_n([nt])$ is a square integrable martingale, and the quadratic variation is $\sum_{k=1}^{[nt]} w_n^2(k)$. On the other hand, it follows from conditions (D1) and (E1) that

$$\frac{1}{n^2}\sum_{k=1}^{[nt]} \mathbf{E}[y_n(k-1)] = \omega_n \int_0^{[nt]/n} \int_0^{[ns]/n} m_n^{[nu]} duds, \tag{4.6}$$

which tends to $\omega \int_0^t \int_0^s e^{au} duds$, as $n\to\infty$. Then applying Doob's inequality to martingale terms in (4.5), we have for sufficiently large n,

$$\begin{aligned}
\mathbf{E}\Big[\sup_{0\le s\le t} Y_n(s)\Big] \le\; & c_n \mathbf{E}\Big[\sum_{k=1}^{[nt]} \sum_{j=1}^{y_n(k-1)} \big(\hat{\xi}_n(k,j) - \chi(\hat{\xi}_n(k,j))\big)\Big] + 2c_n \mathbf{E}^{\frac{1}{2}}\Big[W_n^2([nt])\Big] \\
& + c_n \sum_{k=1}^{[nt]} \mathbf{E}\big[y_n(k-1)\big]\big|\mathbf{E}\big[\chi(\hat{\xi}_n(k,j))\big]\big| + n\omega_n t \\
\le\; & n^2 c_n \Phi(t) \int_\Gamma (u-\chi(u))\mu_n(du) + 2c_n\Big(n^2\Phi(t)\,\mathbf{var}\,\chi(\hat{\xi}_n(1,1))\Big)^{\frac{1}{2}} \\
& + n^2 c_n \Phi(t)\Big|\int_\Gamma \chi(u)\mu_n(du)\Big| + n\omega_n t.
\end{aligned}$$

By (i), (ii), (C), (D1) and (E1), we obtain $nc_n \int_\Gamma \chi(u)\mu_n(du) \to a$ and then (4.3) holds. By (4.5), the sequence $Z_n(\cdot)$ are given by

$$Z_n(t) = \sum_{k=1}^{[nt]} (m_n-1) Z_n\Big(\frac{k-1}{n}\Big) + \sum_{k=1}^{[nt]} \sum_{j=1}^{y_n(k-1)} \frac{\xi_n(k,j)-m_n}{c_n} + \sum_{k=1}^{[nt]} \frac{\eta_n(k)-\omega_n}{c_n}. \tag{4.7}$$

Let $\hat{\eta}_n(k) = \eta_n(k)/c_n$. By Doob's inequality, it is not hard to see that for sufficiently large n,

$$\begin{aligned}
\mathbf{E}\Big[\sup_{0\le s\le t} |Z_n(s)|\Big] \le\; & n|m_n-1| \int_0^t \mathbf{E}[|Z_n(s)|]ds + 2n^2\Phi(t)\int_\Gamma (u-\chi(u))\mu_n(du) \\
& + 2\Big(n^2\Phi(t)\,\mathbf{var}\,\chi(\hat{\xi}_n(1,1))\Big)^{\frac{1}{2}} + 2nt\int_0^\infty (u-\chi(u))\nu_n(du) \\
& + 2\Big(nt\,\mathbf{var}\,\chi(\hat{\eta}_n(1))\Big)^{\frac{1}{2}}.
\end{aligned} \tag{4.8}$$

By Gronwall's inequality and standard stopping argument, (i), (ii), (C), (D1) and Lemma 4.1 implies

$$\mathbf{E}[|Z_n(t)|] \le 2\big\{K(\Phi(t)+t) + 2\big(K(\Phi(t)+t)\big)^{\frac{1}{2}}\big\}\exp\{(|a|+1)t\}, \tag{4.9}$$

where $K = \sup_n \big(n^2 \int_\Gamma (u - \chi(u) + \chi^2(u))\mu_n(du) + n \int_0^\infty (u - \chi(u) + \chi^2(u))\nu_n(du)\big)$. By the above inequality and (4.8), we obtain (4.4). $\square$

Lemma 4.3 *Let $\phi_n(t) = \mathbf{E}[Y_n(t)]/n$ for $t \geq 0$. Under the conditions of Theorem 2.2, the sequence $(Z_n(\cdot), \phi_n(\cdot))$ is tight in $D([0,\infty), \mathbb{R} \times \mathbb{R}_+)$.*

Proof. By Lemma 4.2, $C(t) := 1 + \limsup_{n\to\infty} \big(\frac{1}{n}\mathbf{E}\big[\sup_{0\leq s\leq t} Y_n(s)\big] + \mathbf{E}\big[\sup_{0\leq s\leq t} |Z_n(s)|\big]\big)$ is a locally bounded function of $t \geq 0$. Then $Z_n(t)$ is a tight sequence of random variables for every $t \geq 0$. Now let $\{\tau_n\}$ be a sequence of stopping times bounded by T and let $\{\delta_n\}$ be a sequence of positive constants such that $\delta_n \to 0$ as $n \to 0$. By Doob's Optional Sampling Theorem, we obtain as in the calculations in (4.8) that for sufficiently large n,

$$
\begin{aligned}
&\mathbf{E}\big[\big|Z_n(\tau_n+\delta_n) - Z_n(\tau_n)\big|\big] \\
\leq\ & 2K \int_0^{\frac{[n\delta_n]+1}{n}} \frac{1}{n}\mathbf{E}[y_n([n\tau_n]+[ns])]\,ds + (|a|+1)\int_0^{\frac{[n\delta_n]+1}{n}} \mathbf{E}[Z_n(\frac{[n\tau_n]+[ns]}{n})] \\
&+ \Big(K\int_0^{\frac{[n\delta_n]+1}{n}} \frac{1}{n}\mathbf{E}[y_n([n\tau_n]+[ns])]\,ds\Big)^{\frac{1}{2}} + 2K\Big(\delta_n + \frac{1}{n}\Big) + \Big(K(\delta_n + \frac{1}{n})\Big)^{\frac{1}{2}} \\
\leq\ & (2K + |a| + 1)\int_0^{\delta_n+\frac{1}{n}} C(T+s)ds + \Big(K\int_0^{\delta_n+\frac{1}{n}} C(T+s)ds\Big)^{\frac{1}{2}} \\
&+ 2K\Big(\delta_n + \frac{1}{n}\Big) + \Big(K(\delta_n + \frac{1}{n})\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Then $Z_n(\cdot)$ is tight in $D([0,\infty),\mathbb{R})$ by the criterion of Aldous [1]. It is easy to see that $\phi_n(t)$ converges to $\phi(t) := \omega\int_0^t e^{as}ds$ in distribution on $D([0,\infty),\mathbb{R}_+)$. By Jocod and Schiryaev [12, Corollary 3.33, P.317], $(Z_n(\cdot), \phi_n(\cdot))$ is tight in $D([0,\infty), \mathbb{R}\times\mathbb{R}_+)$. $\square$

Let $Z(\cdot)$ be any limit point of $Z_n(\cdot)$. Without loss of generality, by Skorokhod's theorem, we can assume that on some Skorokhod's space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$, $(Z_n(\cdot), \phi_n(\cdot)) \xrightarrow{a.s.} (Z(\cdot), \phi(\cdot))$ in the topology of $D([0,\infty), \mathbb{R}\times\mathbb{R}_+)$.

Lemma 4.4 *For any fixed $\lambda \in \mathbb{R}$,*

$$
L(t) = e^{i\lambda Z(t)} - e^{i\lambda Z(0)} - \int_0^t e^{i\lambda Z(s)} A(Z(s), \phi(s), \lambda)ds \tag{4.10}
$$

is a complex-valued local $\mathcal{F}_t$-martingale. Here $i^2 = -1$ and

$$
A(x_1, x_2, \lambda) = ia\lambda x_1 + (a\gamma_0\lambda^2/2 - G(-i\lambda))x_2 + \gamma_0(\omega^2 - \omega)\lambda^2/2 - H(-i\lambda),
$$

where G and H are defined by (2.12) and (2.13), respectively.

Proof. Define the stopping times

$$
\begin{aligned}
\tau^b &= \inf\{t \geq 0 : |Z(t)| \geq b \text{ or } |Z(t-)| \geq b\}, \\
\tau_n^b &= \inf\{t \geq 0 : |Z_n(t)| \geq b \text{ or } |Z_n(t-)| \geq b\}.
\end{aligned}
$$

Let $Z^b(t) = Z(t\wedge\tau^b)$, $Z_n^b(t) = Z_n(t\wedge\tau_n^b)$, and analogously $\phi^b(t)$, $\phi_n^b(t)$. It follows from [12, Proposition 2.11, P.305] that for all but countably many b,

$$
\tau_n^b \xrightarrow{a.s.} \tau^b \text{ in } \mathbb{R} \quad \text{and} \quad (Z_n^b(\cdot), \phi_n^b(\cdot)) \xrightarrow{a.s.} (Z^b(\cdot), \phi^b(\cdot))
$$

in the topology of $D([0,\infty),\mathbb{R}\times\mathbb{R}_+)$. Define $\tau_n^b(t)=\tau_n^b\wedge t$ and $\tau^b(t)=\tau^b\wedge t$. We claim that

$$\tau_n^b(\cdot)\xrightarrow{a.s.}\tau^b(\cdot)\quad\text{in } C([0,\infty),\mathbb{R}_+),\qquad\text{as } n\to\infty. \tag{4.11}$$

In fact, since $0\le\tau_n^b(t+\varepsilon)-\tau_n^b(t)\le\varepsilon$ for any $t\ge0$, the criterion of Aldous yields tightness for $\{\tau_n^a(\cdot),\ n\ge1\}$. On the other hand, $\{Z_n(\frac{k}{n}):k\ge1\}$ is a time-inhomogeneous Markov chain. For fixed $\lambda\in\mathbb{R}$,

$$L_n(l)=e^{i\lambda Z_n(\frac{l}{n})}-e^{i\lambda Z_n(0)}-\sum_{k=0}^{l-1}\Big(\mathbf{E}\big[e^{i\lambda Z_n(\frac{k+1}{n})}\big|\mathcal{F}_k^n\big]-e^{i\lambda Z_n(\frac{k}{n})}\Big)$$

is a complex-valued martingale. (1.2) implies that

$$L_n([nt])=e^{i\lambda Z_n(t)}-e^{i\lambda Z_n(0)}-\int_0^{\frac{[nt]}{n}}e^{i\lambda Z_n(s)}n\big[A_n\big(Z_n(s),\phi_n(s),\lambda\big)-1\big]ds, \tag{4.12}$$

where $A_n(x_1,x_2,\lambda)=e^{-i\lambda/c_n\,(n(m_n-1)x_2+\omega_n)}\big(e^{-i\lambda/c_n}g_n(e^{i\lambda/c_n})\big)^{c_nx_1+nx_2}h_n(e^{i\lambda/c_n})$. For simplicity, we denote $L_n([nt])$ by $L_n(t)$. Then $L_n^b(t):=L_n(t\wedge\tau_n^b)$ is also a complex-valued martingale. It follows from the proof of Lemma 2.3 and Lemma 4.1 that

$$n(e^{-i\lambda/c_n}g_n(e^{i\lambda/c_n})-1)\to0\quad\text{and}\quad n^{\frac12}(h_n(e^{i\lambda/c_n})-1)\to i\omega\gamma_0^{\frac12}\lambda, \tag{4.13}$$

as $n\to\infty$. Then we have for sufficiently large n,

$$\begin{aligned}\ln^{A_n(x_1,x_2,\lambda)}&=i\lambda(m_n-1)x_1+(c_nx_1+nx_2)\int_{\Gamma_n}(e^{i\lambda u}-1-i\lambda u)\mu_n(du)\\&\quad+(c_nx_1+nx_2)I_{1,n}(\lambda)+\int_0^\infty(e^{i\lambda u}-1-i\lambda u)\nu_n(du)\\&\quad-\frac12\big(h_n(e^{i\lambda/c_n})-1\big)^2+I_{2,n}(\lambda),\end{aligned} \tag{4.14}$$

where

$$\begin{aligned}I_{1,n}(\lambda)&=\sum_{j=2}^\infty(-1)^{j-1}\frac{[e^{i\lambda/c_n}g_n(e^{i\lambda/c_n})-1]^j}{j},\\I_{2,n}(\lambda)&=\sum_{j=3}^\infty(-1)^{j-1}\frac{[h_n(e^{i\lambda/c_n})-1]^j}{j}.\end{aligned}$$

Note that $n^2|I_{1,n}(\lambda)|\le\big|n(e^{i\lambda/c_n}g_n(e^{i\lambda/c_n})-1)\big|^2\to0$ and $n|I_{2,n}(\lambda)|\to0$. By (i)-(iii), Lemma 4.1, (4.13) and (4.14), it is not hard to show that $n(A_n(x_1,x_2,\lambda)-1)\to A(x_1,x_2,\lambda)$ locally uniformly on $(x_1,x_2)\in\mathbb{R}\times\mathbb{R}_+$ for fixed λ. As in Ethier and Kurtz [5, Problem 26, P153], we obtain that

$$\int_0^t e^{i\lambda Z_n^b(s)}n\big[A_n\big(Z_n^b(s),\phi_n^b(s),\lambda\big)-1\big]ds\to\int_0^t e^{i\lambda Z^b(s)}A(Z^b(s),\phi^b(s),\lambda)ds$$

in the topology of $C([0,\infty),\mathbb{C})$. Let $L^b(t)=L(t\wedge\tau^b)$. Note that $[nt]/n\to t$ in $C([0,\infty),\mathbb{R}_+)$. By (4.11), [5, Problem 13, P.151] and [12, Proposition 1.23, p.293], we have

$$L_n^b(t)\xrightarrow{a.s.}L^b(t)\quad\text{in } D([0,\infty),\mathbb{C}),\qquad\text{as } n\to\infty. \tag{4.15}$$

Then for almost all $t\ge0$, $L_n^b(t)\xrightarrow{a.s.}L^b(t)$ in $\mathbb{C}$. Fix arbitrary $T>0$. For any $t\le T$, $\big|\int_0^{\tau_n^b(t)}e^{i\lambda Z_n^b(s)}\,Z_n^b(s)ds\big|\le bT$, where the bound holds uniformly in n. Then for almost $t\le T$,

$L_n^b(t) \xrightarrow{L_1} L^b(t)$, as $n \to \infty$. Since $L^b(t)$ is right continuous and bounded for $t \le T$, we have $L^b(t)$ is a martingale. Note that $\tau^b \to \infty$ as $b \to \infty$, $L(t)$ is a local martingale. $\square$

It follows from (4.10) and [12, Theorem 2.42] that $Z(\cdot)$ is a semimartingale and it admits the canonical representation

$$Z(t) \;=\; Z(0) + Z^c(t) + \int_0^t \big(\beta_2 + \beta_1\phi(s) + aZ(s)\big)\,ds + \int_0^t \int_0^\infty u\tilde{J}(ds, du), \qquad (4.16)$$

where $Z(0) = 0$, $Z^c(t)$ is a continuous local martingales with quadratic covariation process $\int_0^t \varrho(s)ds$ with $\varrho(s) = (2\sigma_1 - a\gamma_0)\phi(s) + 2\sigma_2 + \omega(1-\omega)\gamma_0$, and $J(dt, dz)$ is an integer-valued random measure on $(0,\infty)\times\mathbb{R}_+$ with compensator $\hat{J}(dt, du) = \phi(t)dt\mu(du) + dt\nu(du)$, where $\tilde{J}(dt, dz) = J(dt, dz) - \hat{J}(dt, du)$.

Lemma 4.5 *Suppose that the conditions of Theorem 2.2 are satisfied. Then the càdlàg process $Z(\cdot)$ is a weak solution of (2.15).*

Proof. Define the measure $\rho(du, d\zeta) = \mu(du)\iota(d\zeta) + \nu(du)\delta_0(d\zeta)$, where $\iota(d\zeta)$ is the Lebesgue measure on $(0,\infty)$ and $\delta_0(d\zeta)$ is the Dirac measure at $\zeta = 0$. By Ikeda and Watanabe [10, P.84 and P.93], there exists a standard extension of $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ supporting a one-dimensional Brownian motion and a Poisson random measure $N(dt, du, d\zeta)$ on $(0,\infty)\times\mathbb{R}_+^2$ with intensity $ds\rho(du, d\zeta)$ such that $dZ^c(t) = \sqrt{\varrho(t)}dB(t)$, and

$$J((0,t]\times E) = \int_0^t \int_{\mathbb{R}_+^2} 1_E\big(\tilde{\theta}(s,u,\zeta)\big)N(ds, du, d\zeta), \qquad (4.17)$$

for any $E \in \mathfrak{B}(\mathbb{R}_+)$, where $\tilde{\theta}(s,u,\zeta) = u1_{[0,\phi(s)]}(\zeta)$. Set $N_0(ds, du) = N(ds, du, \{0\})$ and set $N_1(ds, du, d\zeta) = N(ds, du, d\zeta)|_{(0,\infty)\times\mathbb{R}_+\times(0,\infty)}$. Then we see that $Z(\cdot)$ is a solution of (2.15). $\square$

Proof of Theorem 2.2 By [10, P.231], the Lipschitz conditions of the equation (2.15) imply its pathwise uniqueness of solutions. Thus Theorem 2.2 follows from Lemma 4.3 and 4.5. $\square$

Proof of Theorem 3.1 By Theorem 2.1 $\big(Y_n(\cdot)/b_n, Y_n'(\cdot)/b_n\big)$ converges weakly to $\big(Y(\cdot), Y'(\cdot)\big)$ on $D([0,\infty), \mathbb{R}_+^2)$, and $\big(Y(\cdot), Y'(\cdot)\big)$ is stochastically continuous. Note that

$$n(\check{m} - m_n) = \frac{Y_n(1)/b_n - n(m_n - 1)\int_0^1 Y_n(t)/b_n\,dt - Y_n'(1)/b_n}{\int_0^1 Y_n(t)/b_n\,dt}.$$

Then we have (3.1) by the continuous mapping theorem. Since F is not bounded, the immigration process $Y'(\cdot)$ is neither a compound Poisson process or a zero process. This implies that $P(Y(t) = 0$ for all $t \in [0,1]) = 0$. $\square$

Lemma 4.6 *Define $u_n(k) = y_n(k) - m_n y_n(k-1) - \omega_n$. Then we have*

$$\frac{1}{nc_n}\sum_{k=1}^{n} u_n^2(k) \xrightarrow{p} 0, \quad \text{as } n \to \infty. \qquad (4.18)$$

Proof. It follows from (2.1) and (3.2) that

$$u_n(k) = \sum_{j=1}^{y_n(k-1)} (\xi_n(k,j) - m_n) + (\eta_n(k) - \omega_n).$$

Recall that $\hat{\xi}_n(k,j)$ and $\hat{\eta}_n(k)$ defined in the proof of Lemma 4.2. Note that $\hat{\xi}_n(k,j)-\chi(\hat{\xi}_n(k,j)) \geq 0$ and $\hat{\eta}_n(k) - \chi(\hat{\eta}_n(k)) \geq 0$. Then we have

$$\begin{aligned}
\frac{1}{nc_n}\sum_{k=1}^n u_n^2(k) \;\leq\; & 6(I_{1,n}^2 + I_{2,n}) + \frac{6c_n}{n}\Big[n^{\frac{3}{2}}\int_\Gamma (\chi(u)-u)\mu_n(du)\Big]^2 \int_0^1 \Big(\frac{Y_n(s)}{n}\Big)^2 ds \\
& + 6c_n\Big(\int_0^\infty (\chi(u)-u)\nu_n(du)\Big)^2 + 6\Big[\sqrt{\frac{c_n}{n}}\sum_{k=1}^n \big(\hat{\eta}_n(k) - \chi(\hat{\eta}_n(k))\big)\Big]^2 \\
& + \frac{6c_n}{n}\sum_{k=1}^n (\chi(\hat{\eta}_n(k)) - \mathbf{E}[\chi(\hat{\eta}_n(k))])^2, \qquad (4.19)
\end{aligned}$$

where

$$\begin{aligned}
I_{1,n} &= \sqrt{\frac{c_n}{n}}\sum_{k=1}^n \sum_{j=1}^{y_n(k-1)} \big[\hat{\xi}_n(k,j) - \chi(\hat{\xi}_n(k,j))\big], \\
I_{2,n} &= \frac{c_n}{n}\sum_{k=1}^n \Big[\sum_{j=1}^{y_n(k-1)} \big(\chi(\hat{\xi}_n(k,j)) - \mathbf{E}[\chi(\hat{\xi}_n(k,j))]\big)\Big]^2.
\end{aligned}$$

We obtain that $\mathbf{E}[I_{1,n}] \leq \sqrt{\frac{c_n}{n}}\Phi(1)K$ and $\mathbf{E}[I_{2,n}] \leq \frac{c_n}{n}\Phi(1)\big[n^2 \,\mathbf{var}\,\chi\big(\hat{\xi}_n(1,1)\big)\big]$ as in the calculations in (4.8). Condition (C) implies that $I_{i,n} \xrightarrow{p} 0$ as $n \to \infty$ for $i = 1, 2$. From (C), Remark 2.3, and (a.1) in the proof of Lemma 2.3, the third term in (4.19) converges in probability to 0 as $n \to \infty$. As in the above proof, we also have that the last three terms converge in probability to 0. Thus (4.19) implies (4.18). $\square$

Proof of Theorem 3.2 First consider the equation (2.15). We obtain as in the calculation in (4.8) and (4.10) that for $0 \leq s \leq t$,

$$\begin{aligned}
\mathbf{E}[|Z(t) - Z(s)|] \;\leq\; & |\beta_2|(t-s) + |\beta_1|\int_s^t \phi(u)du + |a|\int_s^t \hat{\Phi}(u)e^{|a|u}du \\
& + 2\int_s^t \phi(u)du \int_0^\infty (u \wedge u^2)\mu(du) + 2(t-s)\int_0^\infty (u\wedge u^2)\nu(du) \\
& + \Big(\int_s^t \phi(u)du\int_0^\infty (u\wedge u^2)\mu(du)\Big)^{\frac{1}{2}} + \Big(\int_s^t \varrho(u)du\Big)^{\frac{1}{2}}, \qquad (4.20)
\end{aligned}$$

where $\hat{\Phi}(\cdot)$ is some non-decreasing continuous function. Then $Z(\cdot)$ is stochastically continuous. Let $D(Z) := \{t \geq 0 : \mathbf{P}\{Z(t) = Z(t-)\} = 1\}$ and thus $D(Z) = (0,\infty)$. From (3.3) we obtain

$$\frac{n^2}{c_n}(\hat{m}_n - m_n) = \frac{\frac{1}{nc_n}\sum_{k=1}^n y_n(k-1)u_n(k)}{\frac{1}{n^3}\sum_{k=1}^n y_n^2(k-1)} = \frac{D(n)}{Q(n)}. \qquad (4.21)$$

Rewrite $D(n)$ as $D(n) = D_1(n) + \frac{c_n}{n}\sum_{j=2}^3 D_j(n) - D_4(n)$, where

$$\begin{aligned}
D_1(n) &= \frac{1}{nc_n}\sum_{k=1}^n \mathbf{E}[y_n(k-1)]u_n(k), \quad D_2(n) = \frac{n(1-m_n^2)}{2m_n}\int_0^1 Z_n(s)ds, \\
D_3(n) &= \frac{1}{2m_n}Z_n^2(1), \qquad D_4(n) = \frac{1}{2nc_nm_n}\sum_{k=1}^n u_n^2(k). \qquad (4.22)
\end{aligned}$$

Let $M_n(t) = \sum_{k=1}^{[nt]} u_n(k)/c_n$ for $t \geq 0$. The functional $\Psi_n : D([0,\infty),\mathbb{R}) \mapsto \mathbb{R}$ is defined by

$$\Psi_n(x) = \omega_n\int_0^1 (x(1) - x(t))m_n^{[nt]-1}dt - \frac{x(1)}{nc_nm_m}. \qquad (4.23)$$

Then $D_1(n)$ can be rewritten as

$$D_1(n) = \frac{1}{nc_n} \sum_{j=1}^{n-1} m_n^{j-1} \sum_{k=j+1}^{n} u_n(k) = \Psi_n(M_n).$$

If $x_n \to x$ in the topology of $D([0,\infty),\mathbb{R})$, it is easy to see that $|\Psi_n(x_n) - \Psi(x)| \to 0$, where

$$\Psi(x) = \omega \int_0^1 (x(1) - x(t)) e^{at} dt.$$

Note that $M_n(t) = Z_n(t) - \int_0^{[nt]/n} n(m_n - 1) Z_n(s) ds$ by (4.7). It follows from Theorem 2.2 that $M_n(t)$ converges weakly to $M(t) := Z(t) - \int_0^t aZ(s)ds$ on $D([0,\infty),\mathbb{R})$. By Remark 2.3 $Y_n(\cdot)/n$ converges weakly to $\phi(\cdot)$ on $D([0,\infty),\mathbb{R}_+)$, and $\phi(\cdot)$ is a deterministic continuous function. Thus $\big(Y_n(\cdot)/n, Z_n(\cdot), M_n(\cdot)\big)$ converges weakly to $\big(\phi(\cdot), Z(\cdot), M(\cdot)\big)$ on $D([0,\infty),\mathbb{R}_+ \times \mathbb{R}^2)$. By [5, Theorem 7.8, P.131 and Problem 26, P.153] and the continuous mapping theorem, we have $D_1(n) \xrightarrow{d} \Psi(M) = \int_0^1 \phi(s) dM(s)$, $D_2(n) \xrightarrow{d} -\int_0^1 aZ(s)ds$, $D_3(n) \xrightarrow{d} \frac{1}{2} Z^2(1)$ and $Q_n \xrightarrow{p} \int_0^1 \phi(s)ds$, as $n \to \infty$. Then it follows from (C) and Lemma 4.6 that $\frac{c_n}{n} \sum_{j=2}^{3} D_j(n) - D_4(n) \xrightarrow{p} 0$. Hence we obtain (3.5). In a similar way, we also have (3.6). □

Recall that $u_n(k) = y_n(k) - m_n y_n(k-1) - \omega_n$. Let $v_n(k) = u_n^2(k) - \pi_n y_n(k-1) - r_n$ and let $V_n(t) = \sum_{k=1}^{[nt]} v_n(k)$. By (3.7), $V_n(\cdot)$ is a martingale.

Proof of Theorem 3.3 Under the conditions of Corollary 2.2, $Z_n(\cdot)$ is defined by (2.14) with $c_n = \sqrt{n}$, and then $M_n(t) = \sum_{k=1}^{[nt]} u_n(k)/\sqrt{n}$. By Corollary 2.2 and the proof of Theorem 3.2, we have that $(Z_n(\cdot), M_n(\cdot))$ converges weakly to $\big(Z(\cdot), M(\cdot)\big)$ on $D([0,\infty),\mathbb{R}^2)$, where $Z(\cdot)$ is given by (2.16) and $M(t) = Z(t) - \int_0^t aZ(s)ds$. Note that $M_n(\cdot)$ is a square integrable martingale and $\mathbf{E}[M_n^2(t)] = (\pi_n \sum_{k=1}^{[nt]} \mathbf{E}[y_n(k-1)] + [nt] r_n)/n$. Then for $t \geq 0$ and sufficiently large n, $\mathbf{E}[M_n^2(t)] \leq \tilde{\mu}(\mathbb{R}_+) \int_0^t \phi(s)ds + \tilde{\nu}(\mathbb{R}_+)t + 1$. Thus by Kurtz and Protter [15, Theorem 2.7],

$$\Big(Z_n(t), M_n(t), \int_0^t Z_n(s-)dM_n(s)\Big) \to \Big(Z(t), M(t), \int_0^t Z(s-)dM(s)\Big) \tag{4.24}$$

in distribution on $D([0,\infty),\mathbb{R}^3)$. On the other hand, let $\hat{V}_n(t) := V_n(t)/n$.

$$\begin{aligned}\hat{V}_n(t) \quad &= \quad Z_n^2(t) + n(1 - m_n^2) \int_0^{[nt]/n} Z_n^2(s)ds - 2m_n \int_0^t Z_n(s-)dM_n(s) \\ &\quad -n\pi_n \int_0^{[nt]/n} Y_n(s)/n\, ds - [nt] r_n/n.\end{aligned}$$

Still note that $Y_n(\cdot)/n$ converges weakly to $\phi(\cdot)$ on $D([0,\infty),\mathbb{R}_+)$, and $\phi(\cdot)$ is a deterministic continuous function. By (4.24) and the continuous mapping theorem, $(Z_n(\cdot), \hat{V}_n(\cdot))$ converges weakly to $(Z(\cdot), J(\cdot))$ on $D([0,\infty),\mathbb{R}^2)$, where $J(t) = Z^2(t) - 2a \int_0^t Z^2(s)ds - 2\int_0^t Z(s)dM(s) - \tilde{\mu}(\mathbb{R}_+) \int_0^t \phi(s)ds - \tilde{\nu}(\mathbb{R}_+)t$. By Itô's formula, $J(t)$ has also the form (3.10). $\hat{V}_n(t)$ is also a finite variation process. Denote its finite variation by $\int_0^t |dV_n(s)|$. Then for $t \geq 0$ and sufficiently large n, $\mathbf{E}\big[\int_0^t |d\hat{V}_n(s)|\big] \leq 2\tilde{\mu}(\mathbb{R}_+) \int_0^t \phi(s)ds + 2\tilde{\nu}(\mathbb{R}_+)t + 1$. By [15] again, $\int_0^t Y_n(s-)/n\, d\hat{V}_n(s)$ converges weakly to $\int_0^t \phi(s)dJ(s)$ on $D([0,\infty),\mathbb{R})$. We see that

$$n(\hat{\pi}_n - \pi_n) = \frac{\int_0^1 Y_n(s-)/n\, d\hat{V}_n(s) - \hat{V}_n(1) \int_0^1 Y_n(s)/n\, ds}{\int_0^1 (Y_n(s)/n - \int_0^1 Y_n(s)/n\, ds)^2 ds}, \tag{4.25}$$

and $\hat{r}_n - r_n = \hat{V}_n(1) - (\hat{\pi}_n - \pi_n)\int_0^1 Y_n(s)ds$. Note that $J(t)$ and $\int_0^t \phi(s)dJ(s)$ are stochastically continuous. By the continuous mapping theorem, we have (3.9). We write

$$\begin{aligned} u_n(k) - \hat{u}_n(k) &= -[(\hat{m}_n - m_n)y_n(k-1)]^2 - (\hat{\omega}_n - \omega_n)^2 + 2(\hat{m}_n - m_n)y_n(k-1)u_n(k) \\ &\quad + 2(\hat{\omega}_n - \omega_n)u_n(k) - 2(\hat{m}_n - m_n)(\hat{\omega}_n - \omega_n)y_n(k-1). \end{aligned}$$

As in the proof of (4.25), also by Theorem 3.2, we have (3.9) holds for $\tilde{\pi}_n$ and $\tilde{r}_n$. □

By the proof of Theorem 3.3, we see that $\hat{V}_n(t) := V_n(t)/n$ converges weakly to $J(\cdot)$ on $D([0,\infty),\mathbb{R})$, where $J(t)$ is defined by (3.10). However when we turn to the case of Example 2.1, $J(t)$ is degenerate to 0. Then in this case, we need the following lemma.

Lemma 4.7 *Let $\bar{V}_n(t) = V_n(t)/\sqrt{n}$. Under the conditions of Theorem 3.4, $\bar{V}_n(\cdot)$ converges in distribution on $D([0,\infty),\mathbb{R})$ to the process $V(\cdot)$, which is defined by $V(t) = \int_0^t \sqrt{\varrho_2(s)}dW(s)$, where $\varrho_2(t) = 2\pi^2\phi^2(t) + (a_4 + 4\pi r)\phi(t) + (b_4 - r^2)$ and $W(\cdot)$ is a one-dimensional Brownian motion.*

Proof. Under the above conditions, we see that $Y_n(\cdot)/n$ converges weakly to $\phi(\cdot)$ on $D([0,\infty),\mathbb{R}_+)$ by Remark 2.3 (ii). Then for any $t \geq 0$,

$$\frac{1}{n}\sum_{k=1}^{[nt]} \mathbf{E}[v_n^2(k)|\mathcal{F}_{k-1}^n] = \frac{1}{n}\sum_{k=1}^{[nt]}\Big((a_{4,n} + 4\pi_n r_n - 3\pi_n^2)y_n(k-1) + 2\pi_n^2 y_n(k-1)^2 + b_{4,n} - r_n^2\Big),$$

which converges in probability to $\int_0^t \varrho_2(s)ds$. Now by the martingale central limit theorem, it suffices to prove that, for any $\varepsilon > 0$ and $t \geq 0$,

$$\frac{1}{n}\sum_{k=1}^{[nt]} \mathbf{E}[v_n^2(k)1_{\{|v_n(k)|>\sqrt{n}\varepsilon\}}|\mathcal{F}_{k-1}^n] \xrightarrow{p} 0, \quad \text{as } n \to \infty. \tag{4.26}$$

We have that $v_n(k) = A_{n,k} + B_{n,k} + C_{n,k} + D_{n,k}$, where

$$A_{n,k} = \sum_{i=1}^{y_n(k-1)} [(\xi_n(k,i) - m_n)^2 - \pi_n], \quad B_{n,k} = 2\sum_{i=1}^{y_n(k-1)} (\xi_n(k,i) - m_n)(\eta_n(k) - \omega_n),$$

$$C_{n,k} = (\eta_n(k) - \omega_n)^2 - r_n, \quad D_{n,k} = 2\sum_{i<j}^{y_n(k-1)} (\xi_n(k,i) - m_n)(\xi_n(k,j) - m_n).$$

Note that for any pair of random variables $\bar{X}$ and $\bar{Y}$, $\mathbf{E}\big[(\bar{X}+\bar{Y})^2 1_{\{|\bar{X}+\bar{Y}|>\varepsilon\}}\big] \leq 4\Big(\mathbf{E}\big[\bar{X}^2 1_{\{|\bar{X}|>\varepsilon/2\}}\big] + \mathbf{E}\big[\bar{Y}^2 1_{\{|\bar{Y}|>\varepsilon/2\}}\big]\Big)$. Thus it suffices to show that (4.26) with $v_n(k)$ replaced by $A_{n,k}$, $B_{n,k}$, $C_{n,k}$, and $D_{n,k}$. Let $\xi_n'(k,i) = (\xi_n(k,i) - m_n)^2 - \pi_n$. As in the proof of [11, Theorem 2.2], we obtain

$$\begin{aligned} &\frac{1}{n}\sum_{k=1}^{[nt]} \mathbf{E}[A_n^2(k)1_{\{|A_n(k)|>\sqrt{n}\varepsilon\}}|\mathcal{F}_{k-1}^n] \\ \leq\ & n\mathbf{E}\big[(\xi_n'(1,1))^2 1_{\{|\xi_n'(1,1)|>\sqrt{n}\varepsilon/2\}}\big]\sum_{k=1}^{[nt]} y_n(k-1)/n^2 + 4n(a_{4,n} - \pi_n^2)^2\varepsilon^{-2}\sum_{k=1}^{[nt]} y_n^2(k-1)/n^3 \\ &+ \sqrt{2}n(a_{4,n} - \pi_n^2)^{\frac{3}{2}}\varepsilon^{-1}\sum_{k=1}^{[nt]} y_n^{\frac{3}{2}}(k-1)/n^{\frac{5}{2}}. \end{aligned}$$

For large enough n, $\pi_n \le \sqrt{n}\varepsilon/2$, and

$$\begin{aligned}\mathbf{E}\big[(\xi_n'(1,1))^2 1_{\{|\xi_n'(1,1)|>\sqrt{n}\varepsilon\}}\big] &\le \mathbf{E}\big[(\xi_n(1,1)-m_n)^4 \mathbf{1}_{\{(\xi_n(1,1)-m_n)^2>\sqrt{n}\,\varepsilon/2\}}\big] \\ &\quad +2(\pi_n a_{4,n}+2\pi_n^3)/\sqrt{n}\varepsilon.\end{aligned}$$

Then, by conditions (c.1,2), (4.26) holds with $v_n(k)$ replaced by $A_{n,k}$. Also by condition (c.3), (4.26) holds for $C_{n,k}$. Let $\bar{\xi}_n(k,i) = \xi_n(k,i) - m_n$. For $D_{n,k}$, we note that

$$\begin{aligned}D_{n,k}^2/4 &= \pi_n \sum_{j=2}^{y_n(k-1)} (j-1)\xi_n'(k,j) + \pi_n \sum_{i=1}^{y_n(k-1)-1} (y_n(k-1)-i)\xi_n'(k,i) \\ &\quad + \sum_{i<j}^{y_n(k-1)} \xi_n'(k,i)\xi_n'(k,j) + 2\sum_{l<i<j}^{y_n(k-1)} (\bar{\xi}_n(k,l))^2\bar{\xi}_n(k,i)\bar{\xi}_n(k,j) \\ &\quad +2\sum_{l<i<j}^{y_n(k-1)} \bar{\xi}_n(k,l)(\bar{\xi}_n(k,i))^2\bar{\xi}_n(k,j) + 2\sum_{l<i<j}^{y_n(k-1)} \bar{\xi}_n(k,l)\bar{\xi}_n(k,i))(\bar{\xi}_n(k,j))^2 \\ &\quad +6\sum_{l<i<j<p}^{y_n(k-1)} \bar{\xi}_n(k,l)\bar{\xi}_n(k,i)\bar{\xi}_n(k,j)\bar{\xi}_n(k,p) + y_n(k-1)(y_n(k-1)-1)\pi_n^2/2.\end{aligned}$$

Then it follows from the above equality that

$$\mathbf{E}[D_{n,k}^4|\mathcal{F}_{k-1}^n] \le 16a_{4,n}^2 y_n^2(k-1) + 416a_{4,n}\pi_n^2 y_n^3(k-1) + 772\pi_n^4 y_n^4(k-1). \tag{4.27}$$

Thus, for any $t \ge 0$, $\frac{1}{n}\sum_{k=1}^{[nt]} \mathbf{E}[D_n^2(k)1_{\{|D_n(k)|>\sqrt{n}\varepsilon\}}|\mathcal{F}_{k-1}^n] \le \frac{1}{n^2\varepsilon^2}\sum_{k=1}^{[nt]} \mathbf{E}[D_n^4(k)|\mathcal{F}_{k-1}^n]$, which converges in probability to 0 by (4.27). In a similar way, we can also prove that (4.26) holds with $v_n(k)$ replaced by $B_{n,k}$. □

Proof of Theorem 3.4 It is not hard to see that for any $t \ge 0$,

$$\begin{aligned}\frac{1}{n^3}\sum_{k=1}^{[nt]} E[y_n^2(k-1)] &= \frac{\pi_n + 2\omega_n m_n}{m_n^2}\int_0^{\frac{[nt]}{n}} m_n^{2[ns]}\int_0^{\frac{[ns]}{n}} m_n^{-2[nu]}\int_0^{\frac{[nu]}{n}} m^{[n\zeta]}d\zeta\ du\, ds \\ &\quad + \frac{r_n+\omega_n^2}{n}\int_0^{\frac{[nt]}{n}}\int_0^{\frac{[ns]}{n}} m^{2[nu]}du\, ds,\end{aligned}$$

which converges to $\int_0^t \phi^2(s)ds$ as $n \to \infty$. Also by (4.6) and the proof of Lemma 4.3, we see that $\bar{V}_n(\cdot)$ is a square integrable martingale and $\sup_n \mathbf{E}[\bar{V}_n^2(t)] < \infty$ for $t \ge 0$. By [15, Theorem 2.7], $\big(Y_n(t)/n, \bar{V}_n(t), \int_0^t Y_n(s-)/n\, d\bar{V}_n(s)\big) \to \big(\phi(t), V(t), \int_0^t \phi(s)dV(s)\big)$ in distribution on $D([0,\infty), \mathbb{R}_+ \times \mathbb{R}^2)$. As in the proof of Theorem 3.3, we have (3.11). □

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